COMPUTER RECREATIONS

How to handle numbers with thousands of digits, and why one might want to

by Fred Gruenberger

EDITOR'S NOTE: The author of "Computer Recreations" this month, Fred Gruenberger, is professor of computer science at California State University at Northridge. Gruenberger's acquaintance with computing machinery began more than 40 years ago; he has since published 28 books on computing, and from 1973 to 1981 he edited the monthly magazine *Popular Computing.*

Beginning next month the "Computer Recreations" department will be conducted by A. K. Dewdney, associate professor of computer science at the University of Western Ontario. Dewdney's chief professional interests are in discrete mathematics and the theory of computation, but he is known to many readers for his investigations of two-dimensional science and technology, described in this space by Martin Gardner (see "Mathematical Games"; SCIENTIFIC AMERICAN, July, 1980). Dewdney's elaboration of this work (*The Planiverse: Computer Contact with a Two-dimensional World*) was recently published by Poseidon Press, and a collection of his essays on topics in computer science, *The Turing Omnibus,* will be published next year by Computer Science Press.

If you have a calculator with a key for squaring a number, try this: enter the number 1.0000001 and press the square key 27 times. The procedure is equivalent to raising the initial number to the 134,217,728th power. The correct result, accurate to 10 significant digits, is 674,530.4707, but the calculator will almost surely give a different answer. The problem is designed to reveal the precision level of the machine. The table on page 24 gives the results obtained with several calculators and with versions of the BASIC and Fortran programming languages running on a few computers. None of the machines gets even seven digits correct.

With most electronic calculators the operation of squaring is not equivalent to entering a number and multiplying it by itself. In the latter operation the factors that enter into the multiplication are limited in their size or precision by the number of digits the machine can display. Squaring, on the other hand, operates on the representation of a number stored in the machine, which generally includes a few "guard digits," that is, extra digits that enter into each calculation but are hidden from the operator. Thus if you calculate the square root of 2 on a machine with an eight-digit display and two guard digits, the result will be shown as 1.4142136 but stored internally as 1.414213562. Pressing the square key should recover the original value 2.0000000, whereas multiplying 1.4142136 by itself gives an answer of 2.000000106.

In most calculations an error in the seventh decimal place is of little consequence. Suppose, however, the calculation is made in a computer program that will execute one sequence of instructions if the value is exactly 2 but a different sequence otherwise; the effect of any inaccuracy could be catastrophic. The safest way of avoiding this hazard is probably to round the calculated value to a known level of precision before testing it for equality with 2. In other circumstances the stratagem of rounding offers no help. In the problem of repeatedly squaring a decimal fraction the only way to improve the quality of the result is to maintain greater accuracy throughout all stages of the calculation.

For most numerical work a precision level of eight or nine digits is ample. It should suffice for balancing a checkbook provided the amounts involved are no greater than $1 million or so. No constant of nature is known with a precision of more than 12 significant digits. Achieving even moderate precision in the final result, however, may call for much higher precision in the course of the calculation. When 1.0000001 is squared 27 times, getting 10 digits correct requires that all calculations along the way be accurate to 15 places.

An example of a calculation in which the need for high precision is absolute is the continuing search for larger prime numbers. For many years, before the development of the electronic digital computer, the largest number known to be prime was $2^{127} - 1$, which has 39 digits in decimal notation. With the aid of the computer, starting in 1952, the record has been broken 16 times; currently it is a number ($2^{132049} - 1$) of 39,751 digits. In testing for primality (that is, in determining whether a number can be divided by any numbers other than 1 and itself) all arithmetic must be exact.

NUMBER OF SQUARINGS	POWER	EIGHT-DIGIT PRECISION	15-DIGIT PRECISION
0	1	1.0000001	1.00000010000000
1	2	1.0000002	1.00000020000001
2	4	1.0000004	1.00000040000006
3	8	1.0000008	1.00000080000028
4	16	1 0000016	1.00000160000120
5	32	1.0000032	1.00000320000496
6	64	1 0000064	1.00000640002016
7	128	1.0000128	1.00001280008128
8	256	1.0000256	1 00002560032640
9	512	1.0000512	1.00005120130818
10	1024	1.0001024	1.00010240523794
11	2048	1.0002048	1.00020482096271
12	4096	1.0004096	1.00040968387705
13	8192	1 0008192	1 00081953559497
14	16384	1.0016391	1.00163974282853
15	32768	1 0032809	1.00328217441361
16	65536	1.0065726	1 00657512149610
17	131072	1 0131884	1.01319347521490
18	262144	1.0265507	1.02656101821804
19	524288	1.0538063	1.05382752412486
20	1048576	1.1105077	1.11055245060312
21	2097152	1.2332274	1.23332674554061
22	4194304	1.5208498	1.52109486126578
23	8388608	2.3129841	2.31372957696917
24	16777216	5.3498954	5.35334455534193
25	33554432	28.621381	28.6582979282091
26	67108864	819.18345	821.298040141993
27	134217728	671061.52	674530.470741078

Result of squaring 1.0000001 repeatedly with limited precision

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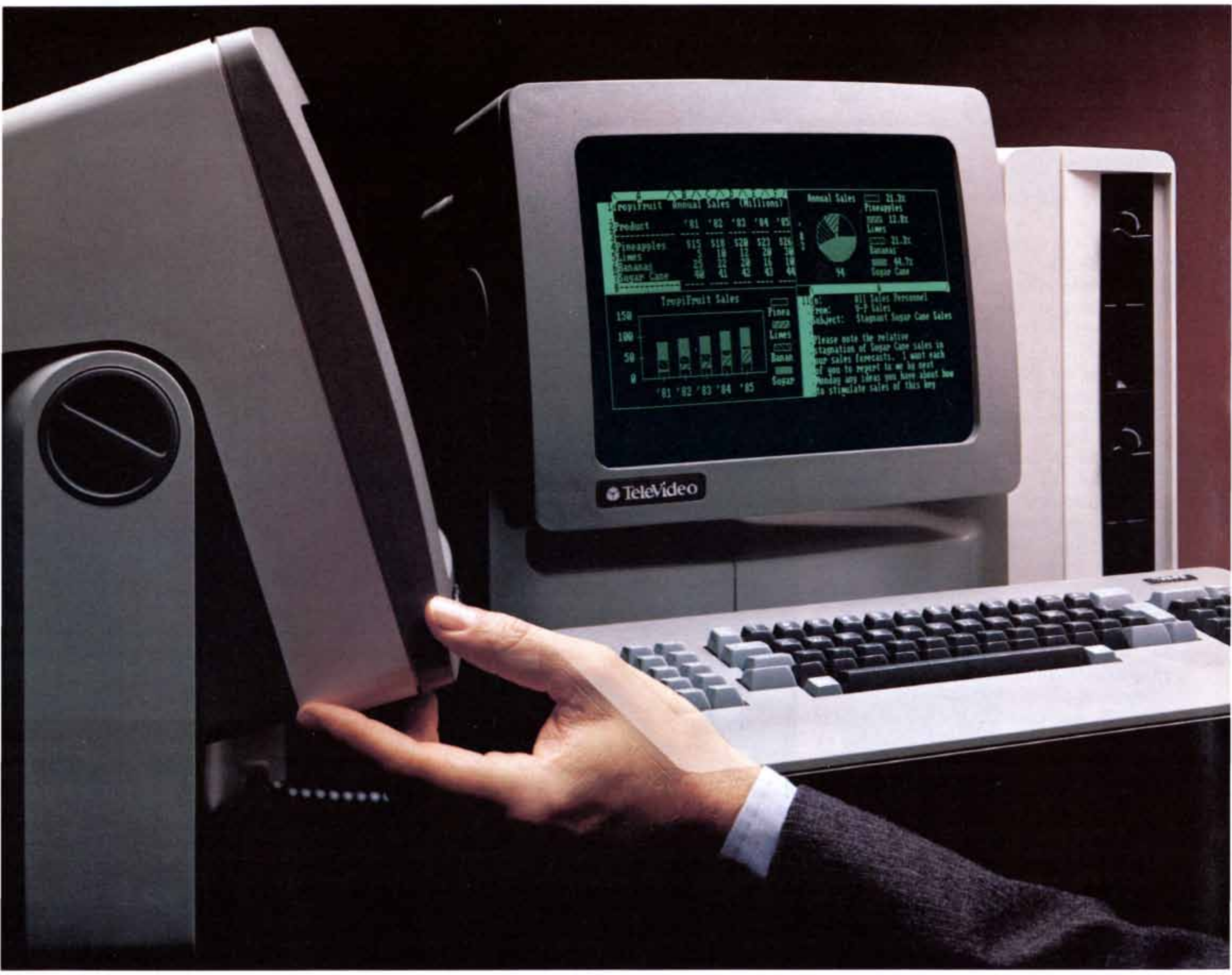

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Tilt Screen	YES	NO	YES	NO
Quiet Operation	YES (NO FAN)	NO	YES	NO
Memory	128K	128K OPTION	256K	256K OPTION
Graphics Display (640x200 resolution)	YES	OPTIONAL	YES	OPTIONAL
Printer Port	YES	OPTIONAL	YES	OPTIONAL
Communication Port	YES	OPTIONAL	YES	YES
MS™-DOS/BASIC®	YES	OPTIONAL	YES	OPTIONAL
System Expansion Slot	YES	YES	YES	YES
RGB and Video Port	YES	OPTIONAL	YES	OPTIONAL
Typical System Price	**$2995**	**$3843**	**$4995**	**$5754**

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Ergonomic Display	YES	NO
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International Power Supply	YES	NO
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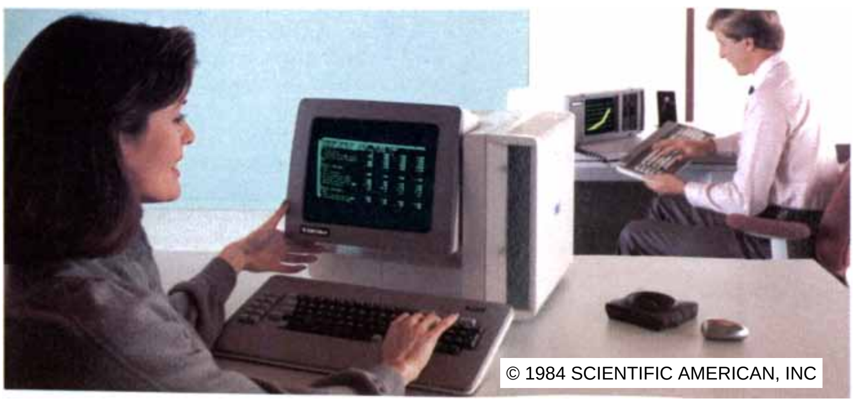

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Computing the value of pi is another problem of long standing that calls for very high precision. The record was held for many years by the British mathematician William Shanks; working by hand, he calculated 528 correct digits (and another 179 incorrect ones). Pi is now known to 8,388,608 digits.

Given that the mechanization of high-precision arithmetic began more than 30 years ago and that computing power has been getting cheaper and becoming more widely available ever since, one might expect the list of known results to be quite extensive. Actually the list of known high-precision numbers is remarkably short:

The square root of 2 is known to one million decimal places, and the cube root of 16 is known to 1,000 places.

The real root of Wallis' equation, $X^3 - 2X - 5 = 0$, is known to 4,000 digits. (The equation is one chosen by the 17th-century British savant John Wallis for an illustration of Newton's method for the numerical solution of equations; it has served since as a test of many other methods of approximation.)

Harry L. Nelson of the Lawrence Livermore National Laboratory (who once held the record with David Slowinski of the same institution for the largest known prime number) has calculated the factorial of one million, namely the product $1{,}000{,}000 \times 999{,}999 \times \ldots \times 2 \times 1$. The result has 5,565,709 digits and fills a stack of standard printout paper five inches high.

A problem known as the 196 problem has been carried through 50,000 stages of calculation, at which point the numbers being dealt with are 21,000 digits long. Another tantalizing problem, the $3N + 1$ problem, has been investigated for isolated values with as many as 1,000 digits. Both problems are discussed in more detail below.

The Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21 . . . , in which each term is the sum of the two preceding terms) has been calculated explicitly for the first 10,000 terms. At the 10,000th term the numbers have more than 2,000 digits.

Euler's number, *e*, the base of the natural logarithms, has been calculated to more than 125,000 decimal places.

A few other isolated results might be cited. For example, R. William Gosper of Symbolics, Inc., employing a method based on the manipulation of continued fractions, has calculated 2,800 digits of the seventh root of 306. (The root has no special significance; the task was selected at random as a test of the method.)

Most of the problems listed above are of the kind in which the need for high precision is intrinsic; they are challenging just because they call for keeping track of more digits than most people can imagine ever needing. A high-precision problem that is commoner and more practical, and less artificial, comes up in solving for the roots of a quadratic equation (an equation that has the form $AX^2 + BX + C = 0$). The quantity $B^2 - 4AC$, called the discriminant of the equation, determines whether the roots are real or imaginary. If the discriminant is positive, the roots are real; if it is negative, they are imaginary; if $B^2 - 4AC$ is exactly zero, the equation has two equal roots. Hence even a small error in the evaluation of the discriminant can make a qualitative difference in the solution of the equation.

A similar sensitivity to small numerical errors can develop in solving a system of simultaneous linear equations. Consider the system

$$
\begin{aligned}
53.17X - 18.91Y - 5.67Z &= -174.65 \\
-12.65X + 36.16Y - 47.08Z &= 298.59 \\
303.80X - 203.03Y + 112.89Z &= -1769.02
\end{aligned}
$$

The system was constructed so that the three equations are satisfied by the values $X = -3$, $Y = 2$ and $Z = -4$, but those values do not represent a unique solution. Indeed, although it is not apparent from mere inspection, there can be no unique solution because two of the equations describe planes that are parallel to each other. (The third equation of the set is equal to five times the first equation minus three times the second equation and so contributes no independent information.)

The impossibility of solving the equations can be discovered by calculating the quantity called the determinant of the matrix of coefficients. The determinant of a matrix is evaluated by forming all possible combinations of elements that are in neither the same column nor the same row; the elements in each combination are multiplied, then the products are summed. If the determinant is found to be zero, the system of equations has no solution and one knows to proceed no further (or, more important, one can arrange to have a computer program halt at this point). Here, however, a calculation of the determinant based on the scheme called basket weaving and using arithmetic accurate to nine digits gives an answer not of zero but of −.000202179. Other methods of evaluation may yield a correct result in this example but not in others; the point is, with nine-digit precision the result cannot be relied on. Moreover, the system of equations given here is a small one, with coefficients having no more than five significant digits; when the system is larger, the problem becomes acute.

Many problems in number theory and other branches of mathematics require extreme precision for the representation of very large integers. The 196 problem is an example. To work the problem start with any positive integer of two digits or more. Reverse the digits and add the reversed number to the original one. Now reverse the sum and add again, and continue the process until the result is a palindromic num-

1.0000001

1.00000020000001

1.0000004000000600000040000001

1.00000080000002800000560000007000000560000028000000080000001

1.00000160000120000056000018200004368000800800114400012870001144000080080
0436800018200000560000012000000160000001

1.000003200004960004960003596002013760906192336585705183028048806451225290
2450257928747373647143565657227801080446572276714356347373622579285290244 8
6451224280488010518300336585609061920201376003596000049600000496000003200 0
0001

1.0000064000201600416640635376762451949744301216634616812205996593289175 61
1024560167423597795591072087154275016633545248630060887595709345161126819 9
1538318569142373529010725957109077232359858708556491166159373356797420116 1
4343435727721880857335556894375111059951058069637780618096826744935403780 3
9908345463896857916128955102413621530314804648290127250191406454777415579 6
9713217570058495461654301216199497436876245120635376004166400020160000064 0
000001

1.0000128000812803413770668026456694236206525938170455867247166397525093 56
3706604966887868632339808586665878774084866169288885995073744765475334516 8
0029637400737469957105560072416700993009294401078544080725549112523177899 2
8499808013093965927574941133244381547633598449089802869082118094468776017 8
2665118692282090047510399909402800454116374069753973388842836767559325442 3
0479144789181357521024292881433616534591710408142167764017463463439708597 2
8390095407066078152415625359663834513241403709468936478003065366991757932 0
3926349554600257842099023918809283454906896251044486185892911937500254243 4
5723208897691093624362353259851586247224033739625021480333630107329307139 2
4088117757151012106967826289242967898516694703304135759892818613724712834 4
6374823047835854926003256454000295386933291445449058223011356249373289045 8
5715015608707021749702661852579574236112264566401066800003413760008128000 0
1280000001

Exact results for a few squarings of 1.0000001

CALCULATOR	RESULT	PERCENT ERROR
Texas Instruments SR-52	674520.6053	.00146
Hewlett-Packard 33, 67, 41C	674494.0561	.00540
Sharp Electronics EL506	674492.75	.00559
Monroe Calculator 1930	674383.1672	.02183
Texas Instruments 30	674363.69	.02473
COMPUTER AND LANGUAGE		
Double-precision Fortran (CDC Cyber)	674530.5363	.00000973
Eight-digit Fortran (CDC Cyber)	674530.5765	.00001568
Apple II BASIC	22723.9709	96.63114
IBM Personal Computer BASIC	8850273.	1212.06423
Ontel BASIC	8886690.	1217.46401

Accuracy of some machines and programming languages in repeated squaring

ber: one that reads the same forward and backward [*see illustration below*]. For most starting numbers a palindrome is reached very quickly; the series beginning with 195, for example, ends after just four steps. The smallest number that seems never to become palindromic by this process is 196; as noted above, it has been tested through 50,000 steps. Among the first 100,000 integers there are 5,996 that apparently do not generate a palindrome no matter how long the procedure is continued (although this conjecture has not been confirmed).

The $3N + 1$ problem was discussed here in January. Start with any positive integer N; if N is odd, replace it with $3N + 1$; if N is even, replace it with $N/2$. Continue until N is equal to 1. For example, when the starting value of N is 9, the process yields 20 terms: 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2 and 1. This simple procedure leads to many mysteries. Does the sequence always terminate at 1, no matter what starting value of N is chosen? Is there any pattern to the number of steps required, that is, can a formula based on the value of N predict the number of steps required? For any chosen number of steps is there invariably an odd value of N that generates a sequence of that length? (An even value of N with any specified path length can readily be found: it is 2 raised to a power of 1 less than the path length.)

In 1980 I proposed that the average number of terms to convergence in the $3N + 1$ problem is approximated by $24.64D - 101$, where D is the number of digits in the starting value of N. The estimate was based on calculations made with numbers of up to about 200 digits, which I then thought were quite large. With a program for high-precision arithmetic I have been able to check the conjecture for a few larger values. For a 1,000-digit number the formula predicts convergence in 25,539 steps. I found that when N is equal to the number 1...(998 zeros)...1, or $10^{1000} + 1$, the series descends to 1 in 23,069 steps. The number 55...(997 zeros)...1 yields a series with 24,413 terms. Hence on this preliminary evidence it appears the average rate of convergence is stable and predictable.

Another phenomenon first observed with small values of N that seems to persist with larger ones is a tendency for many consecutive values of N to generate series of the same length. Indeed, such strings apparently become more prevalent as N increases. For example, the 230 consecutive integers beginning with

9123456789123456789OO-
00000000000000000000001

all generate series with 997 terms. (Note that the prediction of the empirical formula for this number is 959 terms.)

The arithmetic operations that can be carried out directly by the central-processing unit of a typical microcomputer offer only very limited precision. In many machines the only operations provided for explicitly are addition and subtraction of integers whose length is no more than 16 bits, or binary digits; in decimal notation the largest number that can be represented is 65,536, so that the level of precision is less than five decimal digits. Even with the most powerful computers precision of more than a few dozen digits can be attained only by means of a program that combines many elementary operations to break a large number down into smaller pieces.

The flow charts on page 26 outline an algorithm for a specific high-precision calculation: the evaluation of the largest known prime number, $2^{132049} - 1$. The procedure was created with a particular microprocessor in mind, namely the 6502, manufactured by MOS Technology, which is found in computers made by Apple Computer Inc., Commodore Business Machines, Inc., and other manufacturers. It would be easy to adapt the algorithm to other microprocessors.

Perhaps the most fundamental deci-

193	194	195	196	197	198	199	200
391	491	591	691	791	891	991	002
1124	685	786	887	988	1089	1190	202
4211	586	687	788	889	9801	0911	
5335	1271	1473	1675	1877	10890	2101	
	1721	3741	5761	7781	09801	1012	
	2992	5214	7436	9658	20691	3113	
		4125	6347	8569	19602		
		9339	13783	18227	40293		
			38731	72281	39204		
			52514	90508	79497		
			41525	80509			
			94309	171017			
			90349	710171			
			187088	881188			
			880781				
			1067869				
			9687601				
			10755470				
			07455701				
			18211171				
			17111281				
			35322452				
			25422353				
			60744805				
			50844706				
			111589511				
			⋮				

Evaluation of the 196 problem for a few starting values


MULTI-TASKING
MAINTENANCE
MULTI-USER
ELEGANCE
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PROVEN
COMPATIBILITY
HARDWARE INDEPENDENCE
COMPATIBILITY
SOUND INVESTMENT
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PRODUCTIVITY
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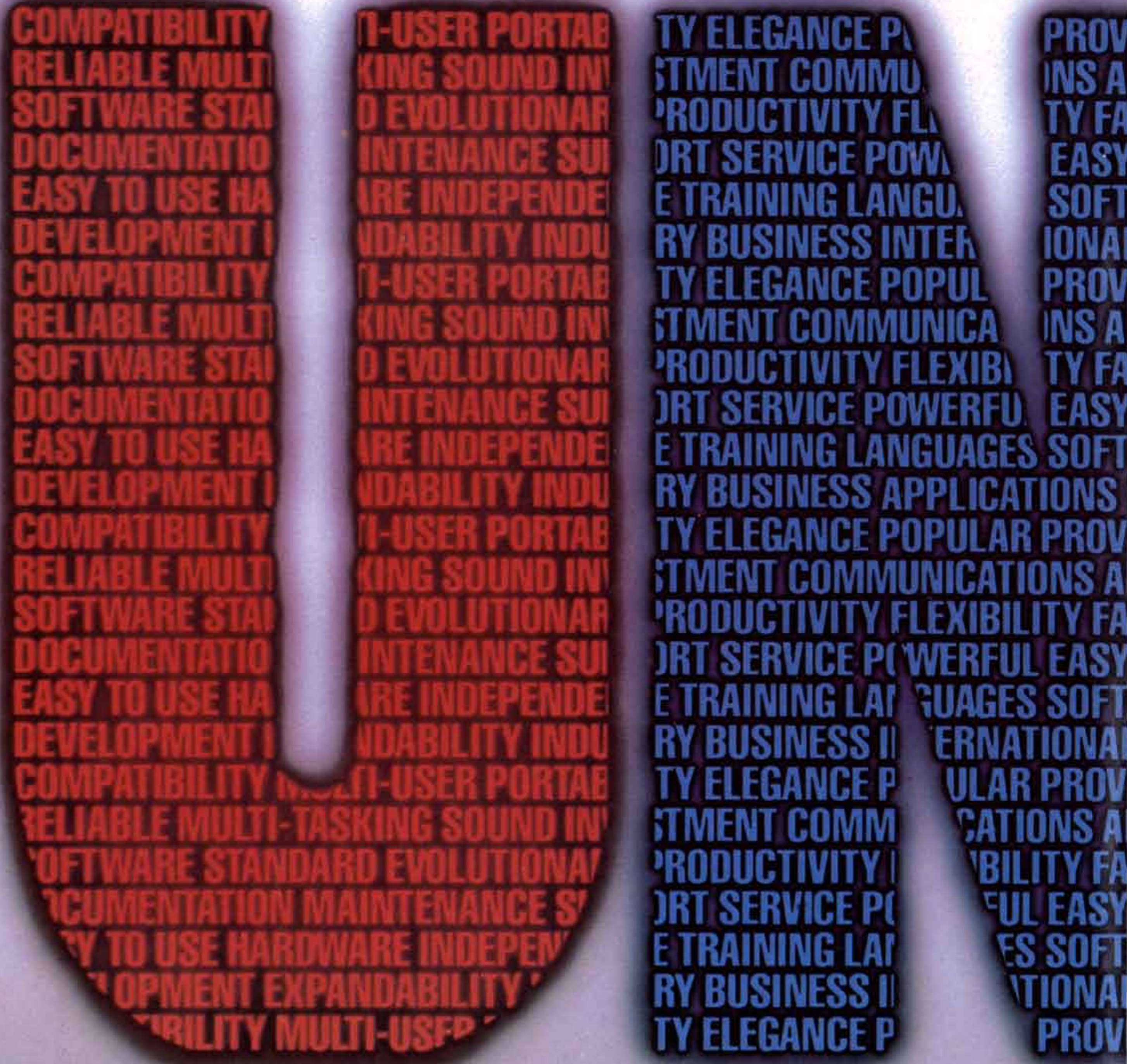
COMPATIBILITY
TRAINING
COMPATIBILITY
PORTABILITY
ELEGANCE
MULTI-USER
STANDARD
HARDWARE INDEPENDENCE
PRODUCTIVITY
MATURE
PROVEN
RELIABLE
DOCUMENTATION
EASY TO USE
DEVELOPMENT
MULTI-TASKING
SOUND INVESTMENT
FLEXIBILITY
EVOLUTIONARY
RELIABLE
SERVICE
EDUCATION
BUSINESS
EXPANDABILITY
MAINTENANCE
DOCUMENTATION
EVOLUTIONARY
DOCUMENTATION
PRODUCTIVITY
FLEXIBILITY
FAMILIARITY
DOCUMENTATION
MAINTENANCE
SUPPORT
SERVICE
MULTI-TASKING
SOUND INVESTMENT
EVOLUTIONARY
SUPPORT
MULTI-TASKING
SOUND INVESTMENT



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Leitz and Leica are registered trademarks of E. Leitz, Inc sion in designing such a program is the choice of how a number is to be represented in the computer's memory. The processor can operate only on binary values, but one would obviously prefer to have the result of the calculation displayed in decimal form. A useful compromise is the scheme called binary-coded decimal, in which each decimal digit is separately represented by its equivalent binary value. The 6502 organizes the memory of a computer in "bytes," or units of eight bits, and so it is convenient to store a number one decimal digit to a byte. (This is not the most efficient method, but it is the simplest.)

The first step in the algorithm is to clear an area in memory 39,760 bytes long by setting each byte equal to zero. Next a starting value of 1 is put into the cleared work space, so that the area holds 39,759 zeros followed by a single 1. A counter that will be needed to keep track of the progress of the calculation is given an initial value of zero.

The main section of the program is a loop that repeatedly calls a subroutine whose function is to double the number stored in the work space. After each doubling the counter is incremented by 1 and the value in the counter is compared with 132,049. If that limit has not yet been reached, the doubling subroutine is called again; when the limit is reached, the program exits from the loop. Finally, 1 is subtracted from the value in the work space and the result is displayed.

In the flow chart at the left the doubling subroutine is marked with color; the instructions making up the subroutine are shown in the flow chart at the right. Each time the routine is called, an index (X) is set to the lowest address in the work space, where the rightmost, or least significant, digit of the number is stored. The value stored at that address is then doubled by adding it to itself. As in doing addition by hand, most of the complication arises from the need to deal with a "carry" from one digit to the next as the addition proceeds. If an earlier addition has generated a carry digit, it must be added to the new result. That result in turn must be checked for a carry digit: if the sum is greater than 9, it must be adjusted by subtracting 10, and the carry digit must be set equal to 1. The process is repeated for all 39,760 bytes of the work space.

The 6502 processor of the Apple II computer operates at a speed of more than 250,000 instructions per second. Nevertheless, the scheme shown in the flow charts would require 120 hours to compute the 132,049th power of 2. Simple short cuts could greatly reduce the running time. For example, it is not necessary to double all the digits of the work space during the early stages of the calculation, when all but a few of them are zeros. The work space might begin with a length of, say, 150 bytes and be augmented by three bytes for every 10 powers. This strategy would in itself require the execution of some additional instructions, but the overall effect would be an increase in speed.

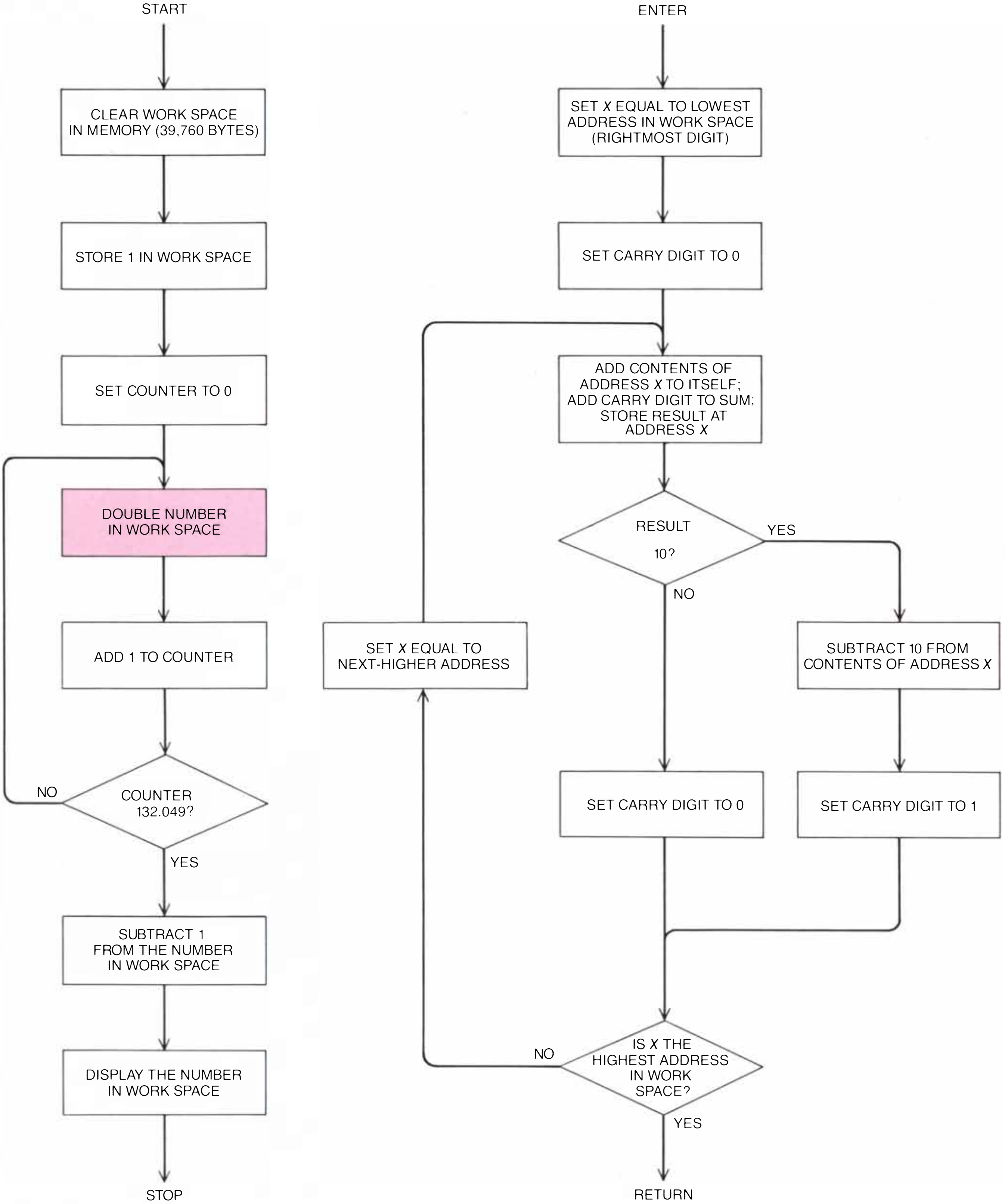
All the numbers in the calculation of $2^{132049} - 1$ are integers, but many problems are best approached by expressing quantities in scientific notation, where a number consists of a decimal fraction called the mantissa and an exponent that gives the magnitude, or power of 10. For example, in the number representing the current year the mantissa is 1.984 and the exponent is $+3$, and the complete number is written as 1.984×10^3. A high-precision program for manipulating such numbers is necessarily more complicated than one that deals only with integers because each number has several parts (including not only the mantissa and the exponent but also their signs).

Most higher-level programming languages incorporate some facilities for calculating in scientific notation; the system is often called floating-point arithmetic. Numbers smaller than a certain size are displayed as an ordinary decimal fraction, but the values are stored internally as a mantissa and an exponent. The space allocated to the various elements of the number determines the precision and the range of values that can be represented. Giving more room to the mantissa improves the precision; a larger exponent affords a greater range. The version of BASIC incorporated into the Apple II has a fixed precision level of about nine digits.

For appreciably greater accuracy it is again necessary to resort to a software solution. Herman P. Robinson, formerly of Lawrence Livermore, has written a package of high-precision scientific-notation programs in the machine language of the 6502 microprocessor. The programs can operate at any level of precision up to 600 decimal digits and allow exponents up to 9,999. These limits were chosen because they match certain characteristics of the processor. A 600-digit mantissa, a four-digit exponent and their signs can all be fitted into 256 bytes; in the 6502 a block of 256 bytes is one "page" of memory.

In Robinson's programs the operations that can be carried out on numbers include the elementary arithmetic ones as well as the logarithmic, exponential, square-root and various trigonometric functions. Some less common functions are also provided, such as the Euler and van Wijngaarden transforms for summing slowly convergent series. Recorded in the package are the values of some 26 constants and 8,000 prime numbers. It can be used as a highly accurate desk calculator, or the functions can be called from within a program. Preliminary versions indicate that the arithmetic operations and the functions are accurate up to the limit of 600 digits.

For those interested in numerical calculations a first acquaintance with the digital computer is sometimes disheartening: because the machine's elementary calculations are fast and essentially flawless, the naive expectation is that elaborate numerical analyses can be done with great ease. One is then disappointed to learn that the fifth root of 100 (the quantity astronomers designate an order of magnitude) cannot be determined with much greater accuracy than a hand-held calculator provides. A package of programs such as Robinson's redeems some of the computer's promise. The fifth root of 100 can be calculated to 100 decimal places in a matter of minutes.



Flow charts for the calculation of the 39,751-digit prime number $2^{132049} - 1$

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We built Chrysler Laser XE to win. We engineered its turbo to endure.
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Laser does it when you equip it with turbo and performance handling package with nitrogen charged shocks.* Laser does it with front wheel drive, dual path suspension system and quick ratio power steering. In the slalom Laser finishes No. 1.

We gave it high-performance braking. Laser XE stops you where Trans Am doesn't.
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We gave it our best: a 5 year/50,000 mile Protection Plan. Even your turbo is protected.
We believe a performer has to be a survivor, so we back your entire powertrain with 5/50 protection; with outer body rust-through protection for the same period.** See dealer for details. Buckle up for safety.

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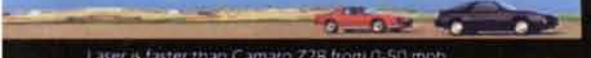

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